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## How to Measure Kinetic Energy of the Heavy Quark Inside B Mesons?

I. Bigi <sup>a,b</sup>, A.G. Grozin <sup>c,d</sup>, M. Shifman <sup>e</sup>, N.G. Uraltsev <sup>a,f</sup>,  
A. Vainshtein <sup>e,d</sup>

<sup>a</sup> *TH Division, CERN, CH-1211 Geneva 23, Switzerland\**

<sup>b</sup> *Dept. of Physics, Univ. of Notre Dame du Lac, Notre Dame, IN 46556, U.S.A.<sup>†</sup>*

<sup>c</sup> *Physics Department, Open University, Milton Keynes MK7 6AA, UK*

<sup>d</sup> *Budker Institute of Nuclear Physics, Novosibirsk 630090, Russia*

<sup>e</sup> *Theoretical Physics Institute, Univ. of Minnesota, Minneapolis, MN 55455*

<sup>f</sup> *St. Petersburg Nuclear Physics Institute, Gatchina, St. Petersburg 188350,  
Russia<sup>†</sup>*

e-mail addresses:

*bigi@cernvm.cern.ch, a.grozin@open.ac.uk, shifman@vx.cis.umn.edu,  
vainshte@vx.cis.umn.edu*

## Abstract

We discuss how one can determine the average kinetic energy of the heavy quark inside heavy mesons from differential distributions in  $B$  decays. A new, so-called third, sum rule for the  $b \rightarrow c$  transition is derived in the small velocity (SV) limit. Using this sum rule and the existing data on the momentum dependence in the  $B \rightarrow D^*$  transition (the slope of the Isgur-Wise function) we obtain a new lower bound on the parameter  $\mu_\pi^2 = (2M_B)^{-1} \langle B | \bar{b}(i\vec{D})^2 b | B \rangle$  proportional to the average kinetic energy of  $b$  quark inside  $B$  meson. The existing data suggest  $\mu_\pi^2 > 0.4 \text{ GeV}^2$  and (from the “optical” sum rule)  $\bar{\Lambda} > 500 \text{ MeV}$ , albeit with some numerical uncertainties.

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\*During the academic year 1993/94

<sup>†</sup>Permanent address

1. In two recent papers [1, 2] it was shown how the operator product expansion (OPE) allows one to derive various useful sum rules for heavy flavor transitions in the small velocity (SV) limit [3]. Non-perturbative corrections are included in the theoretical side of the sum rules in the form of an expansion in the inverse powers of the heavy quark mass. In Ref. [2] the so called first sum rule at zero recoil was obtained which was then used for estimating the deviation of the  $B \rightarrow D^*$  transition form factor from unity at zero recoil to order  $\mathcal{O}(\Lambda_{\text{QCD}}^2)$ . Another sum rule analyzed in [1] yields a field-theoretic proof of the inequality

$$\mu_\pi^2 > \mu_G^2 \quad (1)$$

where  $\mu_\pi^2$  and  $\mu_G^2$  are related to the kinetic energy and the chromomagnetic operators,

$$\mu_\pi^2 = \frac{1}{2M_B} \langle B | \bar{b} (i\vec{D})^2 b | B \rangle, \quad \mu_G^2 = \frac{1}{2M_B} \langle B | \bar{b} (i/2) \sigma G b | B \rangle. \quad (2)$$

(Previously this inequality was obtained within a quantum-mechanical approach [4, 5].) In the present paper we exploit similar ideas to get a new sum rule in the SV limit which relates  $\mu_\pi^2$  to the average square of the excitation energy of the final hadronic state  $X_c$  in the  $B \rightarrow X_c$  semileptonic transitions. At present the corresponding inclusive differential distribution is not yet measured. However, we use the existing data on  $B \rightarrow D^* l \nu$  decays near zero recoil to get a lower bound on  $\mu_\pi^2$ , with no reference to  $\mu_G^2$ . The bound involves the slope of the Isgur-Wise function extracted from the momentum dependence of the  $B \rightarrow D^*$  transition. Numerically this bound turns out to be close to that of Eq. (1).

2. The general method allowing one to derive the sum rules in the SV limit is presented in Ref. [1]. Here we remind only some basic points primarily for the purpose of introducing relevant notations. Operator product expansion is applied to the transition operator [6, 7]

$$\hat{T}_{ab}(q) = i \int d^4x e^{iqx} T \{ j_a^\dagger(x) j_b(0) \} \quad (3)$$

where  $j_a$  denotes a current of the type  $\bar{c} \Gamma_a b$  with an arbitrary Dirac matrix  $\Gamma_a$ ;  $q$  is the momentum carried away by the lepton pair. The average of  $\hat{T}_{ab}$  over the heavy hadron state  $H_b$  with momentum  $p_{H_b}$  represents a forward scattering amplitude (the so-called hadronic tensor),

$$h_{ab}(p_{H_b}, q) = \frac{1}{2M_{H_b}} \langle H_b | \hat{T}_{ab} | H_b \rangle. \quad (4)$$

The observable distributions are expressed through the structure functions  $w_{ab}$ ,

$$w_{ab} = (1/i) \text{disc } h_{ab}.$$

In the HQET limit [8], when one neglects  $1/m_b$ ,  $1/m_c$  corrections, the hadronic tensor  $h_{ab}$  is defined by one invariant function  $h$  for any matrix  $\Gamma_a$  in the current  $j_a$ , namely:

$$h_{ab} = C_{ab}h, \quad C_{ab} = \text{Tr} \left[ \frac{1+\not{v}_1}{2} \bar{\Gamma}_a \frac{1+\not{v}_2}{2} \Gamma_b \right]. \quad (5)$$

Here  $\bar{\Gamma}_a = \gamma_0 \Gamma_a^\dagger \gamma_0$  and  $v_{1\mu}$ ,  $v_{2\mu}$  are 4-velocities of the initial and final hadrons,

$$v_{1\mu} = \frac{(p_{H_b})_\mu}{M_{H_b}}, \quad v_{2\mu} = \frac{(p_{H_b} - q)_\mu}{M_{H_c}}, \quad (6)$$

( $M_{H_b}$  and  $M_{H_c}$  can be substituted by  $m_b$  and  $m_c$ , correspondingly, in the leading approximation). The invariant function  $h$  depends on two scalar invariants available in the process, namely  $(v_1 q)$  and  $q^2$ . In what follows we will assume that the hadron  $H_b$  is at rest; the first invariant then reduces to  $q_0$ . Moreover, in studying the transitions  $b \rightarrow c$  at zero recoil or in the small velocity (SV) limit, it is convenient to employ directly the spacelike momentum transfer  $\vec{q}^2 = (v_1 q)^2 - q^2$  as the second argument of  $h$ .

Taking into account higher order in  $1/m_{b,c}$  we loose, generally speaking, this property of the factorization of  $h_{ab}$  into a universal kinematical structure times one hadronic function  $h$ . In the general case, the hadronic tensor can be decomposed in terms of possible covariants [7] (their number depends on the Lorentz structure of the currents) with coefficients  $h_i$ . In particular, in the case of the vector and axial-vector currents we deal with the functions  $h_i^{VV}$ ,  $h_i^{AA}$  and  $h_i^{VA}$ ,  $i = 1, \dots, 5$  introduced in Ref. [9]. They are independent functions. However, in the leading order they all are expressible in terms of  $h$ , namely,

$$h = \frac{h_1^{AA}}{1 + v_1 v_2} = \frac{1}{2} \frac{m_c}{m_b} h_2^{AA} = -m_c h_5^{AA};$$

$$h = \frac{h_1^{VV}}{1 - v_1 v_2} = -\frac{1}{2} \frac{m_c}{m_b} h_2^{VV} = m_c h_5^{VV};$$

and

$$h = -m_c h_3^{VA}.$$

The functions  $h_i$  not listed here vanish in this approximation. The expressions for all functions  $h_i$  up to order  $1/m_b^2$  can be found in [9].

Although the universal factorization above is not valid for all non-perturbative corrections it still holds for those corrections that are relevant for the third sum rule to be derived below. We will explain this point shortly. Since it is not important what hadronic function we deal with – they all lead to one and the same third sum rule – we will use  $h_1^{AA}$  in our derivation. Thus, we consider the transitions of the  $B$  meson induced by the axial-vector current,

$$A_\mu = \bar{c} \gamma_\mu \gamma_5 b.$$

To single out  $h_1^{AA}$  one must consider the spatial components of the axial current generating the transitions of the  $B$  meson to  $D^*$  and the corresponding excitations.

In [2] the sum rules at zero recoil ( $\vec{q} = 0$ ) were obtained; here we will work at small but non-vanishing values of  $|\vec{q}|$ . The terms  $\mathcal{O}(\vec{q}^2)$  will be kept while those of higher order in  $|\vec{q}|$  will be neglected.

To start the derivation we consider  $h_1^{AA}(q_0, \vec{q})$  in the complex  $q_0$  plane ( $\vec{q}$  is assumed to be fixed, and  $\Lambda_{\text{QCD}} \ll |\vec{q}| \ll M_D$ ). More exactly, let us shift  $q_0$  by introducing the quantity

$$\epsilon = q_{0\max} - q_0 \quad (7)$$

where

$$q_{0\max} = M_B - E_{D^*}, \quad E_{D^*} = M_{D^*} + \frac{\vec{q}^2}{2M_{D^*}}. \quad (8)$$

When  $\epsilon$  is real and positive we are on the cut. The imaginary part of  $h_1$  is given by the “elastic” contribution of  $D^*$  plus inelastic excitations. For what follows it is crucial that all these contributions are *positive definite*.

For negative  $\epsilon$  we are below the cut, and the amplitude  $h_1^{AA}$  can be computed – and it actually was [9, 10, 11] – as an expansion in  $\Lambda_{\text{QCD}}/m_{c,b}$ . For our purposes it is sufficient to limit ourselves to the correction terms of the first and the second order in  $\Lambda_{\text{QCD}}$ . This is exactly the approximation adopted in [9, 10, 11], and expressions obtained there will be used below.

At the next stage we assume that  $\Lambda_{\text{QCD}} \ll |\epsilon| \ll m_{b,c}$ . The amplitude  $h_1^{AA}$  is expanded in powers of  $\Lambda_{\text{QCD}}/\epsilon$  and  $\epsilon/m_{b,c}$ . Polynomials in  $\epsilon$  can be discarded since they have no imaginary part. We are interested only in negative powers of  $\epsilon$ . The coefficients in front of  $1/\epsilon^n$  are related, through dispersion relations, to the integrals over the imaginary part of  $h_1^{AA}$  with the weight functions proportional to the excitation energy to the power  $n - 1$ . Thus, the first sum rule considered in Ref. [2] corresponds to  $n = 1$ ; the second sum rule (sometimes called optical or Voloshin’s sum rule [12], see also [13, 14]) corresponds to  $n = 2$ . The lower bound on  $\mu_\pi^2$  – our main aim in this work – stems from the third sum rule, i.e. we need to analyze the coefficient in front of  $1/\epsilon^3$  in the expansion of  $h_1$ .

The  $1/\epsilon$  expansion can be read off from Eq. (A.1) in Ref. [9]. One technical element of the derivation deserves a comment. The theoretical expression for the amplitude  $h_1^{AA}$  presented in [9] knows nothing, of course, about the meson masses; it contains only the quark masses. Correspondingly, it is convenient to build first the expansion of  $h_1^{AA}$  in an auxiliary quantity,

$$\epsilon_q = m_b - E_c - q_0, \quad E_c = m_c + \frac{\vec{q}^2}{2m_c}. \quad (9)$$

Then, if necessary, we reexpress the expansion obtained in this way in terms of  $\epsilon$ . The difference between  $\epsilon_q$  and  $\epsilon$  is  $\mathcal{O}(\Lambda_{\text{QCD}} \cdot \vec{q}^2/m_{b,c}^2)$  and  $\mathcal{O}(\Lambda_{\text{QCD}}^2/m_{b,c})$ . It will be seen shortly that to our accuracy this difference can be simply ignored in the third sum rule in the SV limit. It can not be discarded, however, in the second sum

rule. (The situation is quite different from that which took place in the sum rules at zero recoil, see [1]. At zero recoil the difference between  $\epsilon$  and  $\epsilon_q$  was absolutely important.)

The expression for  $h_1^{AA}$  in Eq. (A.1) in [9] has the form

$$-h_1^{AA} = [(m_b + m_c - q_0) + \mathcal{O}(\Lambda_{\text{QCD}}^2/m_b)] \frac{1}{z} + \mathcal{O}(\Lambda_{\text{QCD}}^2) \frac{1}{z^2} + \frac{4}{3} (m_b + m_c - q_0) \mu_\pi^2 \bar{q}^2 \frac{1}{z^3} \quad (10)$$

where

$$z = \epsilon_q(2E_c + \epsilon_q). \quad (11)$$

Notice the similarity of the coefficient in front of  $1/z^3$  and the leading part of the coefficient in front of  $1/z$ . This is not accidental. The terms  $1/z^3$  appear only as the expansion of the second order in  $\pi q$  of the denominator  $(m_b v - q)^2 - 2q\pi$  (see Ref. [9]) and, therefore, preserve the same universal factorization which was pointed out above in the HQET limit.

Expanding in  $\epsilon_q/2E_c$  we observe that  $1/z^n$  reduces to  $1/\epsilon_q^n$  plus all lower powers of  $1/\epsilon_q$  plus a polynomial in  $\epsilon_q$ . The next step is eliminating  $\epsilon_q$  in favor of  $\epsilon$ . The term  $1/z^3$  comes with a coefficient  $\mu_\pi^2 \cdot \bar{q}^2$ ; hence here the difference between  $1/\epsilon$  and  $1/\epsilon_q$  is of higher order and can be neglected. By the same token to order  $\mathcal{O}(\Lambda_{\text{QCD}}^2)$  one can substitute  $1/\epsilon_q$  by  $1/\epsilon$  in  $1/z^2$ . As far as  $1/z$  is concerned here we must reexpress  $1/\epsilon_q$  in terms of  $1/\epsilon$ ,

$$\frac{1}{\epsilon_q} = \frac{1}{\epsilon} + \frac{(\epsilon - \epsilon_q)}{\epsilon^2} + \dots \quad (12)$$

Next terms in Eq. (12) are irrelevant since they lead to corrections of higher order in  $\Lambda_{\text{QCD}}$  and/or  $|\vec{q}|$ . This observation is crucial since it tells us that the  $1/z$  part contributes only to the first and the second sum rules; it generates no  $1/\epsilon^3$  terms. As a result  $h_1^{AA}$  has the form

$$\begin{aligned} -h_1^{AA} = & \frac{1}{\epsilon} \left( 1 - \frac{\bar{q}^2}{4m_c^2} + \mathcal{O}(\Lambda_{\text{QCD}}^2/m_c^2) \right) + \frac{1}{\epsilon^2} \left( \mathcal{O}(\Lambda_{\text{QCD}}^2/m_c) + \mathcal{O}(\Lambda_{\text{QCD}} \bar{q}^2/m_c^2) \right) + \\ & \frac{1}{\epsilon^3} \frac{\mu_\pi^2}{3} \frac{\bar{q}^2}{m_c^2} + \text{polynomial}. \end{aligned} \quad (13)$$

Here only the terms  $\mathcal{O}(\bar{q}^2)$  are kept. We also do not discuss perturbative corrections. Writing out the dispersion relation in  $\epsilon$ ,

$$\begin{aligned} -h_1^{AA}(\epsilon, \bar{q}^2) = & \frac{1}{2\pi} \int d\tilde{\epsilon} \frac{w_1^{AA}(\tilde{\epsilon}, \bar{q}^2)}{\epsilon - \tilde{\epsilon}} = \\ & \frac{1}{\epsilon} \cdot \frac{1}{2\pi} \int d\tilde{\epsilon} w_1^{AA}(\tilde{\epsilon}, \bar{q}^2) + \frac{1}{\epsilon^2} \cdot \frac{1}{2\pi} \int d\tilde{\epsilon} \tilde{\epsilon} w_1^{AA}(\tilde{\epsilon}, \bar{q}^2) + \frac{1}{\epsilon^3} \cdot \frac{1}{2\pi} \int d\tilde{\epsilon} \tilde{\epsilon}^2 w_1^{AA}(\tilde{\epsilon}, \bar{q}^2) + \dots \end{aligned} \quad (14)$$

and expanding it in  $1/\epsilon$  we get the sum rules by equating the coefficients in front of  $1/\epsilon^n$ . Here  $w_1^{AA} = 2 \text{Im } h_1^{AA}$ .

3. Now let us discuss the phenomenological side of the sum rule. The structure function  $w_1^{AA}$  is non-vanishing for positive  $\epsilon$ ,

$$w_1^{AA}(\epsilon) = \sum_{i=0}^{\infty} \frac{|F_{B \rightarrow i}|^2}{2E_i} 2\pi\delta(\epsilon - \delta_i), \quad (15)$$

where the sum runs over all possible final hadronic states, the term with  $i = 0$  corresponds to the “elastic” transition  $B \rightarrow D^*$  while  $i = 1, 2, \dots$  represent excited states with the energies  $E_i = M_i + \vec{q}^2/(2M_i)$ . Strictly speaking,  $|F_{B \rightarrow i}|^2$  does not present the square of a form factor; rather this is the contribution to the given structure function coming from the multiplet of the degenerate states which includes summation over spin states as well. In the particular example considered  $D$  is not produced in the elastic transition, so that in the elastic part one needs to sum only over polarization of  $D^*$ . Therefore, the term “form factor” for  $F_{B \rightarrow i}$  is rather symbolic.  $|F_{B \rightarrow i}|^2$  depends on  $\vec{q}$ . Moreover,  $\delta_i$  in Eq. (15) is the excitation energy (including the corresponding kinetic energy),

$$\delta_i = E_i - E_{D^*}.$$

For the elastic transition  $\delta_0$  vanishes, of course.

The dispersion representation (14) and Eq. (13) lead to the following sum rule for the second moment of  $w_1^{AA}$  (the coefficient in front of  $1/\epsilon^3$ , the third sum rule in the nomenclature of Ref. [1]):

$$\frac{1}{2\pi} \int d\epsilon \epsilon^2 w_1^{AA}(\epsilon) = \sum_{i=1}^{\infty} \frac{|F_{B \rightarrow i}|^2}{2E_i} \delta_i^2 = \frac{1}{3} \mu_\pi^2 \frac{\vec{q}^2}{m_c^2}. \quad (16)$$

We pause here to make a few remarks regarding Eq. (16). First of all, since  $\delta_0 = 0$ , this kills the elastic contribution in the left-hand side, and the sum actually starts from the first excitation. Second, since all  $\delta_i^2$  are of order  $\Lambda_{\text{QCD}}^2$  we need to know  $F_{B \rightarrow 2}$ ,  $F_{B \rightarrow 3}$ , etc. only to the zero order in  $\Lambda_{\text{QCD}}$ . To this order all transition form factors to the excited states are proportional to  $\vec{q}$ , i.e.

$$|F_{B \rightarrow i}|^2 \propto \vec{q}^2. \quad (17)$$

(As a matter of fact, the transitions to  $P$ -wave states are relevant, see [15] for further details.) Moreover, taking account of Eq. (17) we can neglect  $\mathcal{O}(\vec{q}^2)$  part in  $\delta_i$ ’s, so that in Eq. (16)

$$\delta_i = M_i - M_{D^*}.$$

Third,  $m_c^{-2}$  in the right-hand side can be replaced, to the accuracy desired, by  $(M_{D^*})^{-2}$  or by the mass of any excited state.

After all these simplifications the third sum rule takes the form

$$\sum_{i=1}^{\infty} \frac{|F_{B \rightarrow i}|^2}{2M_i} (M_i - M_{D^*})^2 = \frac{1}{3} \mu_\pi^2 \vec{v}^2, \quad (18)$$

where

$$\vec{v} = \frac{\vec{q}}{M}$$

(it does not matter whose particular mass,  $M_{D^*}$  or  $M_i$ , stands in the denominator).

The next steps are rather obvious. The lower bound on  $\mu_\pi^2$  is a consequence of positivity of all individual contributions in the left-hand side of Eq. (18). Indeed, let us rewrite it as follows:

$$\frac{1}{3}\mu_\pi^2 = \delta_1^2 \cdot \sum_{i=1}^{\infty} \frac{|F_{B \rightarrow i}|^2}{2M_i \vec{v}^2} + \sum_{i=2}^{\infty} \frac{|F_{B \rightarrow i}|^2}{2M_i \vec{v}^2} (\delta_i^2 - \delta_1^2). \quad (19)$$

The second term is evidently positive. The first sum can be found, in turn, by using the Bjorken sum rule [16]. This sum rule relates the sum over the  $P$ -wave states in the brackets to the  $\vec{q}^2$  dependence of the “elastic”  $B \rightarrow D^*$  transition (the slope of the Isgur-Wise function [17]).

4. It is instructive to briefly reiterate derivation of the Bjorken sum rule, which, as explained above, is needed only in the zero order in  $\Lambda_{\text{QCD}}$ . Equating the coefficients of  $1/\epsilon$  in Eqs. (13) and (14) one immediately finds

$$\frac{1}{2\pi} \int d\epsilon w_1^{AA}(\epsilon) = \sum_{i=0}^{\infty} \frac{|F_{B \rightarrow i}|^2}{2E_i} = 1 - \frac{\vec{v}^2}{4}. \quad (20)$$

The elastic part here can be parametrized in terms of the Isgur-Wise function  $\xi(v_1 v_2)$  [17, 18]. The  $B \rightarrow D^*$  transition has the form

$$\langle D^*(v_2) | A_\mu | B(v_1) \rangle = \sqrt{M_B M_{D^*}} [\epsilon_\mu (1 + v_1 v_2) - (\epsilon v_1) v_{2\mu}] \xi(v_1 v_2)$$

where  $v_{1,2}$  are the four-velocities. This means that

$$(2E_{D^*})^{-1} |F_{B \rightarrow D^*}|^2 = \frac{M_{D^*}}{E_{D^*}} \left( \frac{1 + v_1 v_2}{2} \right)^2 |\xi(v_1 v_2)|^2 \approx 1 - \rho^2 \vec{v}^2. \quad (21)$$

Here  $\rho^2$  is the slope parameter [16],

$$\xi(v_1 v_2) = 1 - \rho^2 (v_1 v_2 - 1) + \dots = 1 - \rho^2 \frac{\vec{v}^2}{2} + \dots \quad (22)$$

and we used the fact that  $\xi$  at zero recoil is unity [3].

Notice that although we discuss the Bjorken sum rule for the axial current actually it can be derived for arbitrary current  $j_a = \bar{b} \Gamma_a c$ . To the leading order in  $1/m_{b,c}$  the universal factorization (5) takes place for the structure functions  $w_{ab} = 2\text{Im } h_{ab}$ . Moreover, the sum over any HQET degenerate multiplet of states gives

$$\frac{1}{2M_B} \sum_i \langle B | j_a^\dagger | H_c^i \rangle \langle H_c^i | j_b | B \rangle = C_{ab} M_{H_c} \frac{1 + v_1 v_2}{2} |\xi_{H_c}(v_1 v_2)|^2 \quad (23)$$

where  $\xi_{H_c}$  is the Isgur-Wise function for the  $H_c$  multiplet.

At  $\vec{v} = 0$  the sum rule (20) is trivially satisfied since at zero recoil all inelastic form factors vanish, and we are left with the elastic contribution which reduces to unity. The term linear in  $\vec{v}^2$  yields a relation between the slope of  $\xi$  and the inelastic contributions,

$$\rho^2 - \frac{1}{4} = \sum_{i=1}^{\infty} \frac{|F_{B \rightarrow i}|^2}{2M_i \vec{v}^2} \quad (24)$$

Let us remind that the ratio  $|F_{B \rightarrow i}|^2/\vec{v}^2$  has the finite limit at zero recoil. Eq. (24) is the Bjorken sum rule proper [16]. Let us add for completeness that in the notations of Ref. [15], where the  $P$  wave inelastic contributions are written out explicitly, it takes the form

$$\rho^2 - \frac{1}{4} = \sum_{n=1}^{\infty} |\tau_{1/2}^{(n)}(1)|^2 + 2 \sum_{n=1}^{\infty} |\tau_{3/2}^{(n)}(1)|^2$$

(for a simple derivation see Ref. [13]). From these expressions it follows, in particular, that  $\rho^2 > 1/4$ .

Combining Eq. (24) with Eq. (19) we finally arrive at

$$\mu_\pi^2 = 3\delta_1^2 \left( \rho^2 - \frac{1}{4} \right) + 3 \sum_{i=2}^{\infty} \frac{|F_{B \rightarrow i}|^2}{2M_i \vec{v}^2} (\delta_i^2 - \delta_1^2), \quad (25)$$

$$\delta_i = M_i - M_{D^*}.$$

Eq. (25) is a direct  $n = 3$  generalization of Voloshin's sum rule written for  $n = 2$  [12], see also [1],

$$\bar{\Lambda} = 2\delta_1 \left( \rho^2 - \frac{1}{4} \right) + 2 \sum_{i=2}^{\infty} \frac{|F_{B \rightarrow i}|^2}{2M_i \vec{v}^2} (\delta_i - \delta_1). \quad (26)$$

Since the second term in Eq. (25) is positive we get the following obvious inequality:

$$\mu_\pi^2 > 3\delta_1^2 \left( \rho^2 - \frac{1}{4} \right) \quad (27)$$

(we remind that  $\delta_1$  here is the lowest excitation energy,  $\delta_1 = M_1 - M_{D^*}$ ). For a first, rough, estimate let us assume that  $\delta_1 \approx 500$  MeV and use the central value of the measured slope [19] of the  $B \rightarrow D^*$  form factor for  $\rho^2$ ,

$$\rho^2 = 0.84 \pm 0.12 \pm 0.08. \quad (28)$$

Then

$$\mu_\pi^2 > 0.45 \text{ GeV}^2.$$

Three comments are in order here regarding the sum rules presented above. First, the very same final results are obtained irrespectively of what currents we start from, axial or vector, or a mixture of these two. The only difference is that, say, for the vector currents we would get  $M_D$ , not  $M_{D^*}$  in the definition of  $\delta_1$ . This difference is unimportant in the limit  $m_{b,c} \rightarrow \infty$ , of course. This remark brings us to



the second point. In Eq. (27) all subleading  $1/m_{b,c}$  terms have been omitted; thus, all quantities there refer to the infinite mass (static) limit, and the corresponding hadronic parameters must be understood just in this sense. In other words, rather than using the experimental value of  $\rho^2$  and  $\delta_1$  measured in the beauty-to-charm transitions and the charmed family, respectively, one should use the static values of  $\rho^2$  and  $\delta_1$ . (Hence,  $M_D$  is indeed equal to  $M_{D^*}$  with our accuracy.) Finally, in the original sum rules the sum runs over all states including those which represent high-energy excitations described, in the dual sense, by perturbative formulae (see Ref. [1] for more details). To get predictions for  $\mu_\pi^2$  and  $\bar{\Lambda}$  normalized at a low (quark-mass independent) scale  $\mu$  one must truncate the sum over the excited states at  $\delta_i \sim \mu$  and invoke duality between the perturbative corrections and the contributions of the excited states above  $\mu$ .

In general, the sum rules at non-zero recoil get  $\Lambda_{\text{QCD}}/m_{b,c}$  corrections which depend on the particular choice of the weak current considered and can be sizable. However, all corrections to the hadronic tensor  $h_{ab}$  start with terms explicitly proportional to  $\Lambda_{\text{QCD}}^2/m_{b,c}^2$  [7, 20, 21], see Eq. (A.1) in Ref. [9]. The question is where the linear corrections come from? A source of subleading corrections is quite obvious: they appear at the stage when one expresses  $\epsilon_q$  in the theoretical formulae in terms of  $\epsilon$ ; since  $M_B = m_b + \bar{\Lambda} + \dots$  (and the same for the charmed quark) they contain linear terms. This does not affect, of course, the first sum rule ( $n = 1$ ), and in this case the prediction starts from unity plus corrections at the level  $\bar{\Lambda}^2/m_{b,c}^2$  [2].

5. We proceed now to a more careful discussion of the numerical situation. The experimentally measured  $B \rightarrow D^*(\text{unpolarized}) l\nu$  decay rate is expressed in terms of the Isgur-Wise function in the leading approximation, see Eq. (21). In this approximation the slope of the Isgur-Wise function is related to the  $\bar{q}^2$  dependence of the  $B \rightarrow D^*$  rate. It is clear that with  $1/m_{b,c}$  and radiative corrections included the  $\bar{q}^2$  dependence of the decay rate does not exactly coincide any more with the slope of the Isgur-Wise function. These corrections were estimated in the literature (see the review paper [22]). Their effect seems to be equivalent to a decrease in the slope of the Isgur-Wise function, by about 9%. At the same time the radiative perturbative corrections were found <sup>1</sup> to increase the slope by  $\sim 20\%$ . Therefore, taking these estimates at their face value we are inclined to conclude that

$$\rho^2 \approx \rho_{exp}^2 - 0.1 \quad (29)$$

where  $\rho^2$  in the left-hand side is the genuine static value of the slope parameter while  $\rho_{exp}^2$  is the value of the parameter obtained from experimental fits of the dependence of the  $B \rightarrow D^*(\text{unpolarized}) l\nu$  decay rate on the  $D^*$  velocity. It is worth emphasizing again that the slope  $\rho^2$  of the Isgur-Wise function by definition does not depend on the structure of weak currents considered. The above numerical

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<sup>1</sup>Although the sign of the perturbative effects is obvious on general physical grounds, we think that the concrete procedure of evaluating them described in Ref. [22] systematically overestimates the velocity dependence, at least for the observable we discuss here.

estimates of both, perturbative and  $1/m_{b,c}$  corrections, have been obtained for the real  $V - A$  current to which experimental numbers refer.

Similar effects due to the finite mass of the  $c$  quark enter our lower bound implicitly when we use the observed mass values of the excited charmed mesons. In the future these pre-asymptotic corrections can be isolated in a model-independent way once the masses of the beauty counterparts are measured. The most sizable corrections are expected due to the chromomagnetic interaction of the heavy quark spin inducing hyperfine splitting among the members of the heavy spin multiplets. In particular,  $M_{D^*} - M_D \sim 140$  MeV. This effect is presumably accounted for by substituting the spin averaged masses for the ground  $S$ -wave states and for the  $P$ -wave excitations, rather than actual masses of  $D$ ,  $D^*$ , etc. We actually did this spin averaging. Another shift arises due to the heavy quark kinetic energy term in the hadron mass. It is natural to expect its value to be smaller in the excited mesons than for the ground state. Therefore, the static limit of  $\delta_1$  is expected to be somewhat larger than the value of  $\delta_1$  experimentally observed for the actual charmed particles, but probably not more than by 50 MeV. We then use the value

$$\delta_1 \approx 500 \text{ MeV} \quad (30)$$

as a very reasonable educated guess for the static value of  $\delta_1$ .

With the parameters from Eqs. (29), (30) we finally get

$$\begin{aligned} \mu_\pi^2 &> 0.37 \text{ GeV}^2, \\ \bar{\Lambda} &> 500 \text{ MeV} \end{aligned} \quad (31)$$

where the second relation comes from Voloshin's sum rule (26). These lower bounds are seen to lie not very far from the estimates obtained earlier within QCD sum rules [23]

$$\mu_\pi^2 \sim 0.55 \text{ GeV}^2, \quad \bar{\Lambda} \sim 450 \text{ MeV}. \quad (32)$$

Note that the lower bound on  $\mu_\pi^2$  in Eq. (31) is numerically quite close to the bound (1) derived recently in [4, 5].

Unfortunately, numerical uncertainties in all the numbers above prevent us from making too strong a statement. Nevertheless, let us assume for a moment that future refined measurements and calculations of the subleading corrections in the third sum rule will confirm these values and establish the fact that two inequalities in Eq. (31) are rather close to saturation. This would mean that the sum rules are actually saturated – to a reasonable degree of accuracy – by the contributions from the states with masses around  $M_D + \delta_1$  generically called  $D^{**}$  in this context. To account for nonperturbative effects in  $b \rightarrow c$  decays one then would need only to consider one inelastic channel, “ $D^{**}$ ”. The higher excited states will be represented (in the sense of duality) by purely perturbative probabilities calculated in the free quark-gluon approximation. We actually consider such a situation as a most natural scenario in QCD. It is worth noting that the  $D\pi$  contribution to the third sum rule

is suppressed for soft pions, unlike the first sum rule, where it was quite substantial [2]. The effect of the “hard” pion emission is well represented by some of the  $P$ -wave  $D^{**}$  resonances.

6. We have derived the third sum rule for the  $b \rightarrow c$  transition in the SV limit and showed how one can use it to constrain the kinetic energy parameter  $\mu_\pi^2$  by using the data on  $B \rightarrow D^*$ . In principle it is quite conceivable that the full differential distribution in  $q_0$  and  $\vec{q}^2$  in the inclusive semileptonic  $B$  decays will be measured in the future. This measurement can then be immediately translated in the value of  $\mu_\pi^2$ , one of the most important parameters of the heavy quark physics. The more one will learn about the decays to the excited states the more accurate the determination of  $\mu_\pi^2$  will become.

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